Lecture2: Mathematical Model of Mechanical Systems

1. Introduction:

There are two types of mechanical systems based on the type of motion:

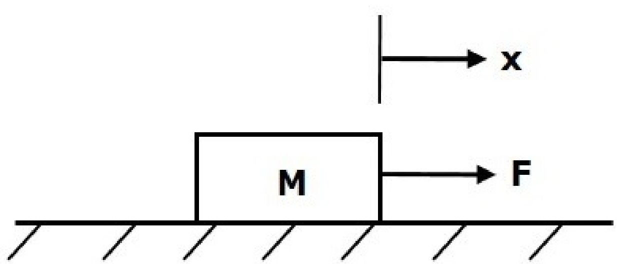
- Translational mechanical systems
- Rotational mechanical systems

A. Modeling of Translational Mechanical Systems:

Translational mechanical systems move along a **straight line**. These systems mainly consist of three basic elements. Those are mass, spring and dashpot or damper (a device the decreases the oscillations of a system).

If a force is applied to a translational mechanical system, then it is opposed by opposing forces due to mass, elasticity and friction of the system. Since the applied force and the opposing forces are in opposite directions, the algebraic sum of the forces acting on the system is zero. Let us now see the force opposed by these three elements individually.

1. **Mass** is the property of a body, which stores **kinetic energy**. If a force is applied to a body having mass **M**, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and friction are negligible.



$$F_m \propto a$$

$$\Rightarrow F_m = M\,a = M\frac{d^2x}{dt^2}$$

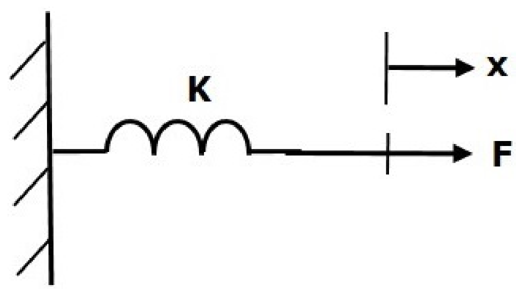
$$F = F_m = M\frac{d^2x}{dt^2}$$

Where,

- **F** is the applied force

- $\mathbf{F_m}$ is the opposing force due to mass
- **M** is mass
- ***a*** is acceleration
- ***x*** is displacement

2. **Spring** is an element, which stores **potential energy**. If a force is applied to spring **K**, then it is opposed by an opposing force due to elasticity of spring. This opposing force is proportional to the displacement of the spring. Assume mass and friction are negligible.



$$F \propto x$$

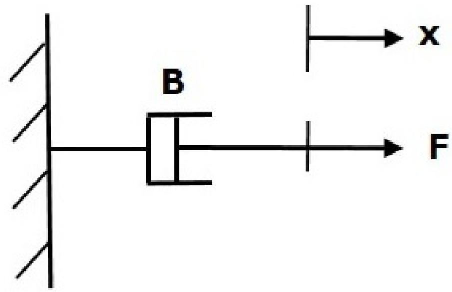
$$\Rightarrow F_k = K\,x$$

$$F = F_k = K\,x$$

Where,

- **F** is the applied force
- $\mathbf{F_k}$ is the opposing force due to elasticity of spring
- **K** is spring constant
- ***x*** is displacement

3. **Dashpot:** A mechanical damping device consisting of a piston that moves through a viscous fluid (usually oil); used, in conjunction with a spring, in shock absorbers. If a force is applied to dashpot **B**, then it is opposed by an opposing force due to **friction** of the dashpot. This opposing force is proportional to the velocity of the body. Assume mass and elasticity are negligible.



$$F_b \propto \dot{x}$$

$$\Rightarrow F_b = B\,\dot{x} = B\frac{dx}{dt}$$

$$F = F_b = B\frac{dx}{dt}$$

Where,

- $\mathbf{F_b}$ is the opposing force due to friction of dashpot
- **B** is the frictional coefficient
- $\dot{\boldsymbol{x}}$ is velocity
- $\boldsymbol{x}$ is displacement

B. Modeling of Rotational Mechanical Systems:

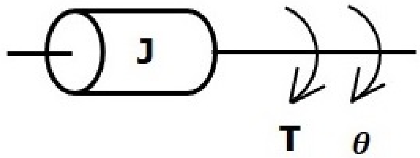
Rotational mechanical systems move about a fixed axis. These systems mainly consist of three basic elements. Those are moment of inertia, torsional spring and dashpot.

If a torque is applied to a rotational mechanical system, then it is opposed by opposing torques due to moment of inertia, elasticity and friction of the system. Since the applied torque and the opposing torques are in opposite directions, the algebraic sum of torques acting on the system is zero. Let us now see the torque opposed by these three elements individually.

1. Moment of Inertia:

In translational mechanical system, mass stores kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores **kinetic energy**.

If a torque is applied to a body having moment of inertia J, then it is opposed by an opposing torque due to the moment of inertia. This opposing torque is proportional to angular acceleration of the body. Assume elasticity and friction are negligible.



$$T_j \propto \alpha$$

$$\Rightarrow T_j = J\,\alpha = J\frac{d^2\theta}{dt^2}$$

$$T = T_j = J\frac{d^2\theta}{dt^2}$$

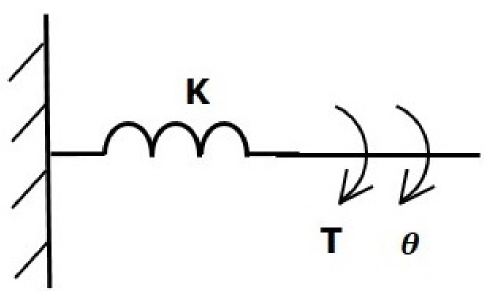
Where,

- **T** is the applied torque
- $\mathbf{T_j}$ is the opposing torque due to moment of inertia
- **J** is moment of inertia
- $\boldsymbol{\alpha}$ is angular acceleration
- **θ** is angular displacement

2. **Torsional Spring**

In translational mechanical system, spring stores potential energy. Similarly, in rotational mechanical system, torsional spring stores **potential energy**.

If a torque is applied to torsional spring **K**, then it is opposed by an opposing torque due to the elasticity of torsional spring. This opposing torque is proportional to the angular displacement of the torsional spring. Assume that the moment of inertia and friction are negligible.



$$T_k \propto \theta$$

$$\Rightarrow T_k = K\,\theta$$

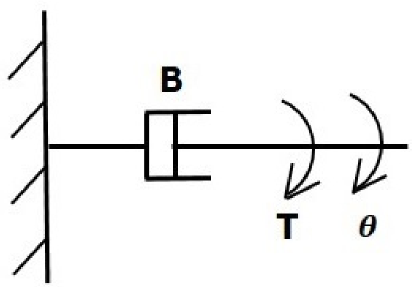
$$T = T_k = K\,\theta$$

Where,

- **T** is the applied torque
- **T_k** is the opposing torque due to elasticity of torsional spring
- **K** is the torsional spring constant
- **θ** is angular displacement

3. Dashpot:

If a torque is applied to dashpot **B**, then it is opposed by an opposing torque due to the **rotational friction** of the dashpot. This opposing torque is proportional to the angular velocity of the body. Assume the moment of inertia and elasticity are negligible.



$$T_b \propto \dot{\theta}$$

$$\Rightarrow T_b = B\,\dot{\theta} = B\,\frac{d\theta}{dt}$$

$$T = T_b = B\,\frac{d\theta}{dt}$$

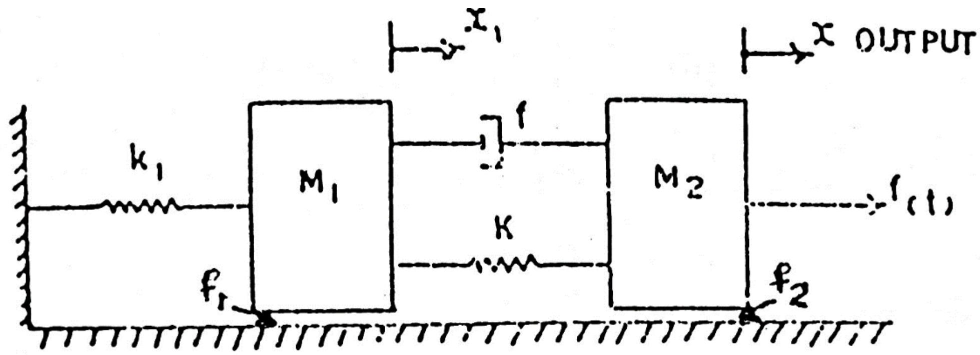
Where,

- **T_b** is the opposing torque due to the rotational friction of the dashpot
- **B** is the rotational friction coefficient
- $\dot{\boldsymbol{\theta}}$ is the angular velocity
- **θ** is the angular displacement

Force-voltage and force-current analogy is very commonly used. The analog quantities between electrical and mechanical systems for these two analogies are tabulated in table below.

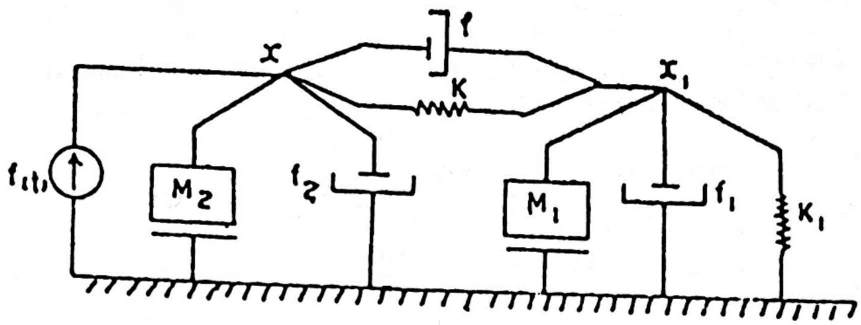
S. No.	Force-Voltage Analog			Force-Current Analog		
	Mechanical System		**Electrical Systems**	**Mechanical System**		**Electrical Systems**
	Translational	**Rotational**		**Translational**	**Rotational**	
1	Force (F)	Torque (T)	Voltage (v)	Force (F)	Torque (T)	Current (i)
2	Mass (M)	Moment of inertia (J)	Inductance (L)	Mass (M)	Moment of inertia (J)	Capacitance (C)
3	Viscous Friction coefficient (B)	Viscous Friction coefficient (B)	Resistance (R)	Viscous Friction coefficient (B)	Viscous Friction coefficient (B)	Reciprocal of resistance (1/R)
4	Spring Stiffness (K)	Torsional spring stiffness (K)	Reciprocal of capacitance (1/C)	Spring Stiffness (K)	Torsional spring stiffness (K)	Reciprocal of inductance (1/L)
5	Displacement (x)	Angular displacement (Θ)	Charge (q)	Displacement (x)	Angular displacement (Θ)	Flux linkages (ϕ)
6	Velocity ($\dot{x}$)	Angular velocity ($\dot{\theta}$)	Current (i)	Velocity ($\dot{x}$)	Angular velocity ($\dot{\theta}$)	Voltage (v)

Example 1: Obtain the transfer function of the mechanical system shown in Figure below and draw it's analog circuit using force-voltage analog.



Solution:

The network diagram of the above mechanical system is shown in Figure below:



Writing the nodal equations at each node, we get:

Node x: $M_2\frac{d^2}{dt^2}x + f_2\frac{d}{dt}x + \mathrm{f}\,(\frac{d}{dt}x - \frac{d}{dt}x_1) + \mathrm{K}(x - x_1) = \mathrm{f(t)}$

Or $(M_2s^2 + f_2s + \mathrm{f\,s} + \mathrm{K})\mathrm{X(s)} - (\mathrm{f\,s} + \mathrm{K})X_1(\mathrm{s}) = \mathrm{F(s)}$ ... (1)

Node x_1: $M_1\frac{d^2}{dt^2}x_1 + f_1\frac{d}{dt}x_1 + \mathrm{K}_1\,x_1 + \mathrm{f}\left(\frac{d}{dt}x_1 - \frac{d}{dt}x\right) + \mathrm{K}(x_1 - x) = 0$

Or $(M_1s^2 + f_1s + \mathrm{f\,s} + \mathrm{K} + K_1)X_1(\mathrm{s}) - (\mathrm{f\,s} + \mathrm{K})\mathrm{X(s)} = 0$ ... (2)

Substituting the value of $\mathrm{X}_1(\mathrm{s})$ from equation (2) in equation (1), we get:

$$(M_2s^2 + f_2s + \mathrm{f\,s} + \mathrm{K})\mathrm{X(s)} - \frac{(\mathrm{f\,s} + \mathrm{K})(\mathrm{f\,s} + \mathrm{K})\,X(s)}{(M_1s^2 + f_1s + \mathrm{f\,s} + \mathrm{K} + K_1)} = F(s)$$

or $[(M_2s^2 + f_2s + \mathrm{f\,s} + \mathrm{K})(M_1s^2 + f_1s + \mathrm{f\,s} + \mathrm{K} + K_1) - (\mathrm{f\,s} + \mathrm{K})^2]X(s) = (M_1s^2 + f_1s + \mathrm{f\,s} + \mathrm{K} + K_1)\,F(s)$

$$Transfer\ function = \frac{X(s)}{F(s)}$$

$$= \frac{(M_1s^2 + f_1s + \mathrm{f\,s} + \mathrm{K} + K_1)}{M_1M_2s^4 + (M_1f_2 + M_2f_1 + M_1f + M_2f)s^3 + (M_2K_1 + K(M_1 + M_2) + f_1f_2 + f(f + f_1 + f_2))s^2 + (K_1(f + f_2) + K(2f + f_1 + f_2))s + K^2 + KK_1}$$

Converting the nodal equations (1) and (2) into comparable electrical analog equations, we get:

R	L	C	f(t)
$v_R = R\,i_R$	$v_L = L\,\frac{d\,i_L}{dt}$	$v_C = \frac{1}{C}\int i_C\,dt$	e(t)

Let f(t) = e(t)

Let $M_2 \frac{d^2}{dt^2} x \approx L_2 \frac{d\,i}{dt}$

$\therefore L_2 = M_2$

$\& \, i \approx \frac{d}{dt} x \Rightarrow x \approx \int i \, dt$

But $f_2 \frac{d}{dt} x \approx R_2 \, i$

$\therefore R_2 = f_2 \; and \; R = f$

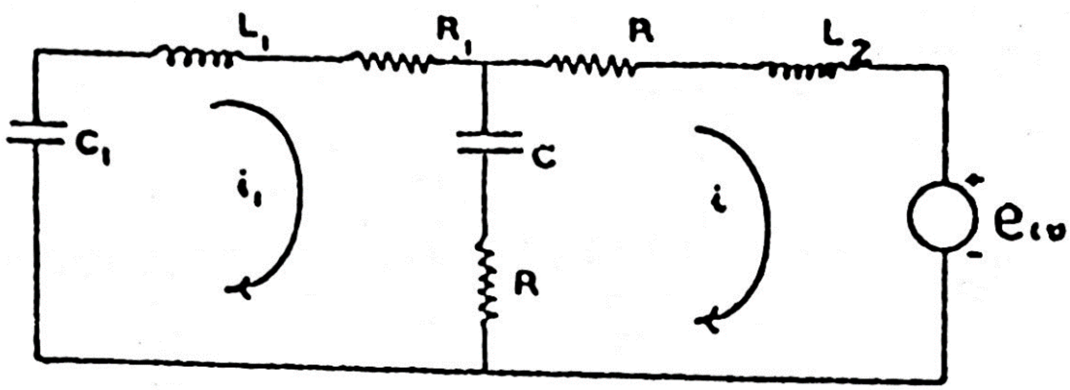
$\mathrm{K} \, x \approx \frac{1}{C} \int i \, dt \;\; and \; \mathrm{K} \, x_1 \approx \frac{1}{C} \int i_1 \, dt$

$\therefore \mathrm{K} = \frac{1}{C}$

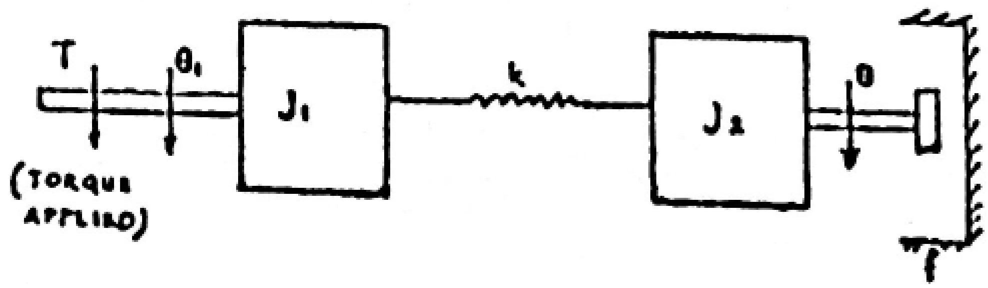
$$L_2 \frac{di}{dt} + R_2 i + R(i - i_1) + \frac{1}{C} \int (i - i_1) dt = e(t) \qquad \text{... (3)}$$

$$L_1 \frac{d\,i_1}{dt} + R_1 i_1 + \frac{1}{C_1} \int i_1 \, dt + R(i_1 - i) + \frac{1}{C} \int (i_1 - i) dt = 0 \qquad \text{... (4)}$$

Based on equations (3) and (4) the electrical analog circuit based on force-voltage analogy in shown below.

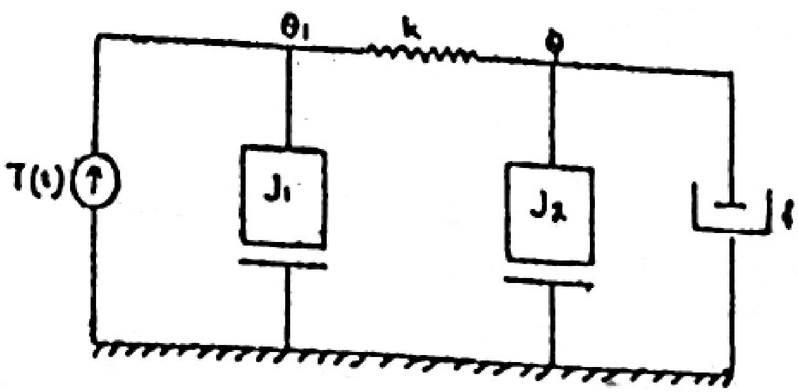


Example 2: Obtain the transfer function of the mechanical system shown in Figure below.



Solution:

The network diagram of the above mechanical system is shown in Figure below:



The nodal equations are:

Node $\boldsymbol{\theta_1}$: $J_1 \frac{d^2}{dt^2}\theta_1 + \mathrm{K}(\theta_1 - \theta) = \mathrm{T(t)}$

Node $\boldsymbol{\theta}$: $J_2 \frac{d^2}{dt^2}\theta + f\frac{d}{dt}\theta = \mathrm{K}(\theta_1 - \theta)$

or $(J_1s^2 + K)\theta_1(s) - K\,\theta(s) = T(s)$ ... (1)

and or $(J_2s^2 + fs + K)\theta(s) = K\,\theta_1(s)$ ... (2)

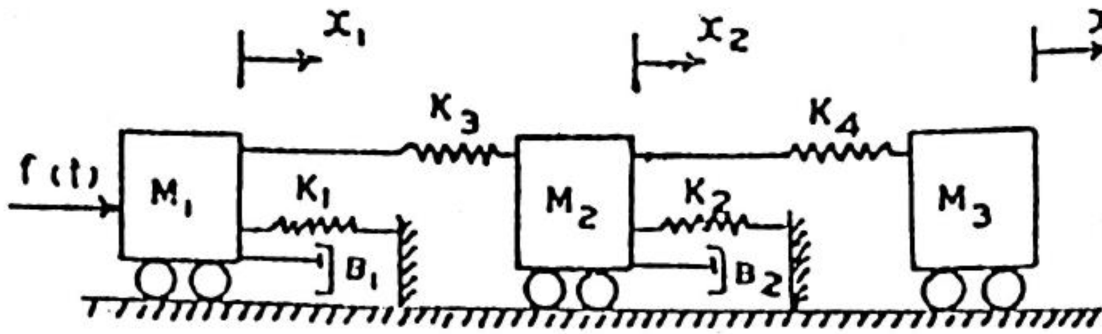
substituting value of $\theta_1(s)$ from equation (2) in equation (1), we get:

$$\left[\frac{(J_1s^2 + K)(J_2s^2 + fs + K)}{K} - \frac{K}{1}\right]\theta(s) = T(s)$$

$$\therefore Transfer\ function = \frac{\theta(s)}{T(s)} = \frac{K}{(J_1s^2 + K)(J_2s^2 + fs + K) - K^2}$$

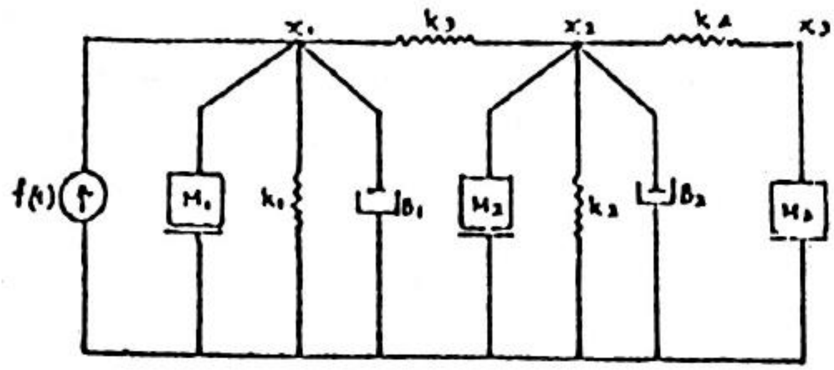
$$= \frac{K}{J_1J_2s^4 + J_1fs^3 + (KJ_1 + KJ_2)s^2 + Kfs}$$

Example 3: Draw the mechanical network for the system in Figure below and draw it's analog circuit using force-current analog.



Solution:

The network diagram of the above system is shown in Figure below:



Node x_1: $f(t) = M_1 \frac{d^2}{dt^2} x_1 + B_1 \frac{d}{dt} x_1 + K_1 x_1 + K_3 x_1 - K_3 x_2$

Node x_2: $K_3 x_1 - K_3 x_2 = M_2 \frac{d^2}{dt^2} x_2 + B_2 \frac{d}{dt} x_2 + K_2 x_2 + K_4 x_2 - K_4 x_3$

or $M_2 \frac{d^2}{dt^2} x_2 + B_2 \frac{d}{dt} x_2 + (K_2 + K_3 + K_4) x_2 - K_3 x_1 - K_4 x_3 = 0$

Node x_3: $M_3 \frac{d^2}{dt^2} x_3 = K_4 x_2 - K_4 x_3$

Using force-current analog:

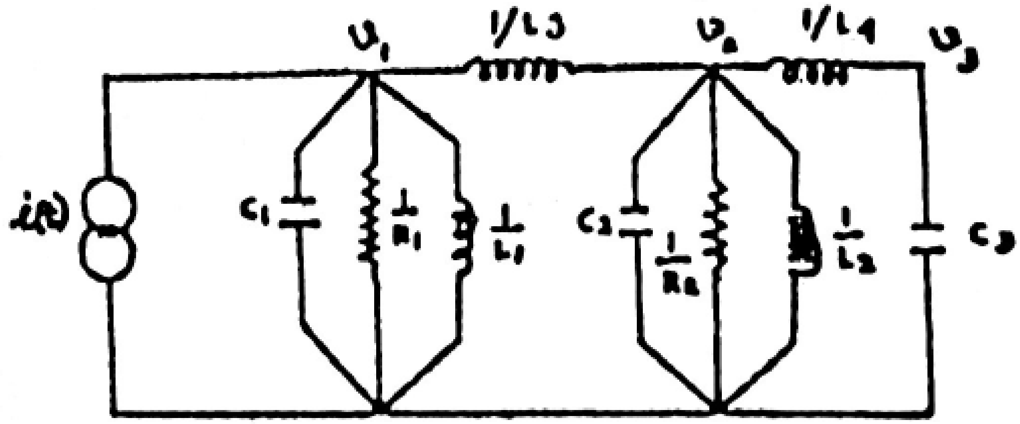
R	L	C	f(t)
$i_R = \frac{v_R}{R}$	$i_L = \frac{1}{L} \int v_L \, dt$	$i_C = C \frac{d\, v_C}{dt}$	$i(t)$

$$i(t) = C_1 \frac{d\, v_1}{dt} + \frac{v_1}{R_1} + \frac{1}{L_1} \int v_1 \, dt + \frac{1}{L_3} \int v_1 \, dt - \frac{1}{L_3} \int v_2 \, dt$$

$$C_2 \frac{d\, v_2}{dt} + \frac{v_2}{R_2} + \left(\frac{1}{L_2} + \frac{1}{L_3} + \frac{1}{L_4}\right) \int v_2 \, dt - \frac{1}{L_3} \int v_1 \, dt - \frac{1}{L_4} \int v_3 \, dt = 0$$

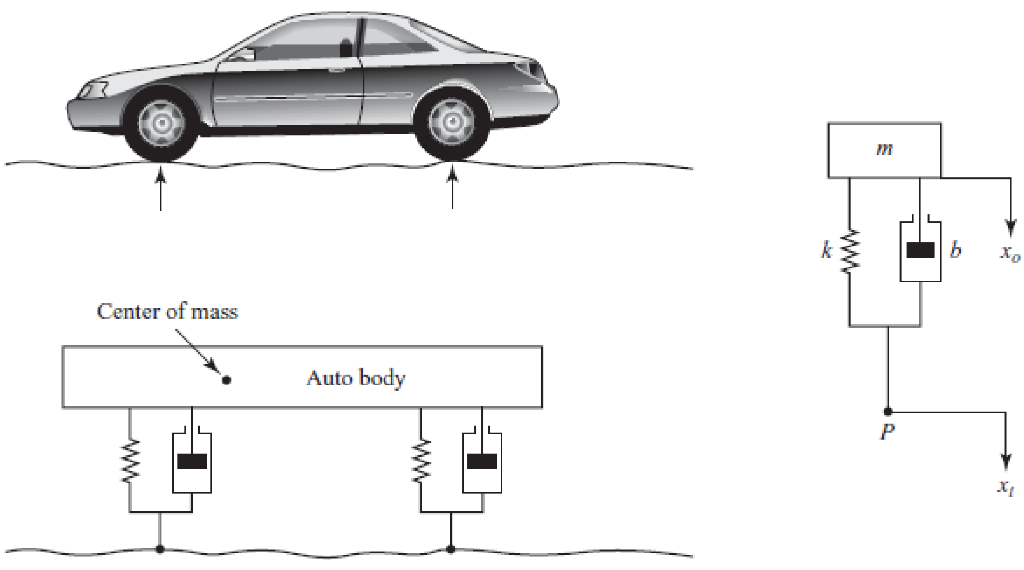
$$C_3 \frac{d\, v_3}{dt} + \frac{1}{L_4} \int v_3\, dt - \frac{1}{L_4} \int v_2\, dt = 0$$

The analog circuit is shown in Figure below:



Example 4: Figure (a) below shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

A very simplified version of the suspension system is shown in Figure (b) below. Assuming that the motion x_i at point ***P*** is the input to the system and the vertical motion x_o of the body is the output, obtain the transfer function $\mathbf{X_o(s)/X_i(s)}$. (Consider the motion of the body only in the vertical direction). Displacement x_o is measured from the equilibrium position in the absence of input x_i.



(a) (b)

Solution. The equation of motion for the system shown in Figure (b) is:

$$m\frac{d^2}{dt^2}x_0 + b\left(\frac{d}{dt}x_0 - \frac{d}{dt}x_i\right) + \mathrm{K}(x_0 - x_i) = 0$$

or $m\frac{d^2}{dt^2}x_0 + b\,\frac{d}{dt}x_0 + Kx_0 = b\,\frac{d}{dt}x_i + \mathrm{K}\,x_i$

Taking Laplace transform, we get:

or $(m\,s^2 + b\,s + K)\,X_0(s) = (b\,s + \mathrm{K})\,X_i(s)$

$$\therefore Transfer\ function = \frac{X_0(s)}{X_i(s)} = \frac{(b\,s + \mathrm{K})}{(m\,s^2 + b\,s + K)}$$